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Homotopy Perturbation Method to Solve Generalized KDV Equation

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ABSTRACT

In this paper, the reliable semi analytical homotopy perturbation method (HPM) is successfully applied to find out the approximate solution of generalized Kortweg-de Vries (KdV), $K(2,2)$ and modified KdV equations that appears in fluid dynamics. This also shows the efficiency of the method.

Key Words: *Partial Differential Equation, Homotopy Perturbation Method, KdV Equation.*

2010 Mathematics Subject Classification: Primary- 37K10, 37K40, 65Nxx.

1. Introduction

In various field of science like physics, chemistry, biology and engineering problems are modeled by linear and non-linear differential equations and to solve these problems many analytical, numerical and perturbation method has been developed. One of them is homotopy perturbation method (HPM), it is an effective tool for solving linear and nonlinear differential equations. It is introduced and further devolved by Ji-Huan [1-4]. It is a semi analytical method which gives series solution with great accuracy. HPM is successfully applied to many ordinary or partial linear, non-linear differential equations to give analytical or approximate solutions [5-12].

In this paper HPM is used to solve the generalized KdV equation. The KdV equation was introduced by Dutch mathematician Diederik Korteweg and Gustav de Vries (1895). It is a mathematical model of waves on shallow water surface. the following equation.

$$u_t - 3(u^2)_x + u_{xxx} = 0, \qquad (1.1)$$

represents KdV equation which gives solitary wave solution. Solitons are non-linear solitary waves with infinite support that maintain its shape while it propagate with constant velocities and also these are considered as a solution of solitary non-linear partial differential equations.

Now we are considering $K(m,n)$ equation defined as

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m,n \geq 1 \qquad (1.2)$$

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This is a generalization of KdV equation, and it reduces to KdV equation by choosing $m = 2, n = 1$. The solution of $K(m,n)$ equation gives compactons. Compactons are solitons with finite wave length or it is also defined as solitons with compact support. Now we are taking modified KdV equation also termed as Burger equation [15].

$$u_t - \frac{1}{2}(u^2)_x - u_{xx} = 0, \tag{1.3}$$

It appears in fluid dynamics and used to describe the structure of shock waves. Previously these equations are solved by many researchers using various techniques like inverse scattering method [13], tanh method [14], variational iteration method [15], homotopy analysis method [16], Adomian decomposition method [17], homotopy perturbation method [18]. Here in this paper, we have generalized all the equations and then used homotopy perturbation method to solve these equations then we have taken some special cases for particular values of θ. We get the given classical KdV's equations by choosing $\theta = \frac{\pi}{2}$.

2. Homotopy Perturbation Method (HPM)

To describe the basic idea of this method considering equation as [1-4]:

$$A(u) = f(r), \ \ r \in \Omega \tag{2.1}$$

subject to the condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \ r \in \Gamma \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal vector drawn outwards from Ω. The operator A can generally be divided into two parts M as linear and N as nonlinear part. Therefore, (2.1) can be rewritten as follows:

$$M(u) + N(u) = f(r), \qquad r \in \Omega. \tag{2.3}$$

A homotopy $v(r,p) : \Omega \times [0,1] \rightarrow R$ is constructed in [1, 2] which satisfies

$$H(v,p) = (1-p)[M(v) - M(u_0)] + p[A(v) - f(r)] = 0, \tag{2.4}$$

which is equivalent to

$$H(v,p) = M(v) - M(u_0) + pM(u0) + p[N(v) - f(r)] = 0, \tag{2.5}$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is an initial approximation of (2.3). We have

$$H(v,0) = M(v) - M(u_0) = 0, \quad H(v,1) = A(v) - f(r) = 0 \tag{2.6}$$

As p changes from 0 to 1, $H(v,p)$ changes from $M(v) - M(u_0)$ to $A(v) - f(r)$. This change in topology, called deformation and $M(v)-M(u_0)$ and $A(v)-f(r)$ are called homotopic. According to the HPM, p is a small parameter, and the solution of (2.4) can be expressed as a series in p in the form

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \quad (2.7)$$

when $p \to 1$, (2.4) becomes given problem (2.1) and (2.7) becomes the approximate solution of (2.1), i.e.,

$$u = \log_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots \quad (2.8)$$

The convergence of the series in (2.8) is discussed in [1,2].

3. Generalized KdV Equations with HPM

Prasad et.al. [19] introduced the generalized differential operator as

$$\Delta_x^* = \left(\frac{\partial}{\partial x} - ixcot\theta\right), \quad \theta \neq n\pi, \quad n \in \mathbb{Z} \quad (3.1)$$

also we have

$$\Delta_x^{*2} = \left(\frac{\partial^2}{\partial x^2} - 2ixcot\theta\frac{\partial}{\partial x} - (icot\theta + x^2\cot^2\theta)\right), \quad (3.2)$$

And

$$\Delta_x^{*3} = \left(\frac{\partial^3}{\partial x^3} - 3ixcot\theta\frac{\partial^2}{\partial x^2} - 3(icot\theta + x^2\cot^2\theta)\frac{\partial}{\partial x} - (3xcot^2\theta - ix^3\cot^3\theta)\right), \quad (3.3)$$

Example 3.1. Considering the generalized linear KdV equation with as

$$u_t - 3\Delta_x^* u^2 = 0, \quad -\infty < x < \infty, \quad t > 0, \quad (3.4)$$

subject to the initial condition

$$u(x,0) = 6x. \quad (3.5)$$

To solve equation (3.4) by using HPM, we construct the following homotopy as

$$(1-p)[v_t - (u_0)_t] + p[v_t - 3\Delta_x^* v^2 + {\Delta_x^*}^3 v = 0]$$

from above we have

$$v_t + p\left[-6v\frac{\partial v}{\partial x} + 3ixv^2cot\theta + {\Delta^*}_x^3 v\right] = 0 \quad (3.6)$$

Substituting (2.7) in (3.6) and equating like powers of p, we get

$$p^0: \ \frac{\partial v_0}{\partial t} = 0 \quad (3.7)$$

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$$p^1 \; : \quad \frac{\partial v_1}{\partial t} - 6v_0 \frac{\partial v_0}{\partial x} + 3ixv_0^2 cot\theta + \Delta^{*3}_{\;x} v_0 = 0 \qquad (3.8)$$

$$p^2 \; : \quad \frac{\partial v_2}{\partial t} - 6v_0 \frac{\partial v_1}{\partial x} + 6ixv_0 v_1 cot\theta + \Delta^{*3}_{\;x} v_0 = 0 \qquad (3.9)$$

and so on. Now solving equation (3.7) using given initial condition, we get

$$(v_0)(x,t) \;=\; 6x, \qquad (3.10)$$

Similarly, from (3.8), we have

$$v_1(x,t) \;=\; t[216x \;+\; 36x^2 \cot^2\theta \;+\; i(18cot\theta \;-\; 108x^3\, cot\theta \;-\; 6x^4 \cot^3\theta\,)] \qquad (3.11)$$

Now from (3.9) we can find v_2 as,

$$v_2(x,t) = \frac{t^2}{2!}[15552x + 6930x^2 \cot^2\theta + 630x^3 cot^4\theta - x^6(108\cot^4\theta + 6\cot^5\theta) + i(1944cot\theta + 630xcot\theta - 15552x^3 cot\theta - 2592x^4\cot^3\theta - 126x^5\cot^5\theta) \qquad (3.12)$$

Similarly we can find v_3,v_4 and so on. The required solution is given as

$$u(x,y) = \; v_0 \;+\; v_1 \;+\; v_2 + \ldots \qquad (3.13)$$

Special case 1: When $\theta = \frac{\pi}{4}$, the required solution of KdV equation is given as:

$$\left[u(x,t) = \frac{6x}{1-36t} + 36tx^2 + \frac{t^2}{2!}(6930x^2 + 630x^3 - 114x^6)\right] + i\left[t(18 - 108x^3 - 6x^4) + \frac{t^2}{2!}(1944 + 630x - 15552x^3 - 2592x^4 - 126x^5)\right] + \ldots$$

Special Case 2:When $\theta = (2n + 1)\pi/2$, where $n \in Z$, then (3.4) is reduced as

$$u_t \;-\; 3(u^2)_x \;+\; u_{xxx} \;=\; 0, \qquad (3.14)$$

defined in [18] with initial condition (3.5), solution is given as

$$u(x,t) = \frac{6x}{1-36t} \qquad . \qquad (3.15)$$

Example 3.2. Now considering generalized $K(2,2)$ equation as

$$u_t + \Delta^{*2}_{\;x} u^2 + \Delta^{*3}_{\;x} u^2 = 0, \quad x \in \mathbb{R}, \quad t > 0 \qquad (3.16)$$

With initial condition

$$u(x,0) = x \qquad (3.17)$$

Proceeding as above constructing homotopy as

$$v_t + p\left[\Delta^{*}_{x} v^2 + \Delta^{*3}_{\;x} v^2\right] = 0 \qquad (3.18)$$

Substituting the value of v in (3.18) and rearranging equation based on powers of p, we get

$$p^0 \quad : (v_0)_t = 0 \tag{3.19}$$

$$p^1 \quad : (v_1)_t + 2v_0 \frac{\partial^3 v_0}{\partial x^3} + 6\left(\frac{\partial v_0}{\partial x} - ixcot\theta v_0\right)\frac{\partial^2 v_0}{\partial x^2} + (2 - 6(icot\theta + x^2 \cot^2\theta))v_0 \frac{\partial v_0}{\partial x} - 6ixcot\theta \left(\frac{\partial v_0}{\partial x}\right)^2 - (ixcot\theta + 3xcot^2\theta - ix^3 \cot^3\theta)v_0^2 = 0 \tag{3.20}$$

$$p^2 \quad : (v_2)_t + 2v_0 \frac{\partial^3 v_1}{\partial x^3} + 2v_1 \frac{\partial^3 v_0}{\partial x^3} + 6\left(\frac{\partial v_0}{\partial x} - ixcot\theta v_0\right)\frac{\partial^2 v_1}{\partial x^2} + 6\left(\frac{\partial v_1}{\partial x} - ixcot\theta v_1\right)\frac{\partial^2 v_0}{\partial x^2} + \{(2 - 6(icot\theta + x^2\cot^2\theta))\} \times \{v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x}\} - 12ixcot\theta \frac{\partial v_0}{\partial x}\frac{\partial v_1}{\partial x} - (ixcot\theta + 3xcot^2\theta - ix^3\cot^3\theta)2v_0 v_1 = 0 \tag{3.21}$$

and so on.

Now, solving (3.19), we get

$$v_0(x,t) = x \tag{3.22}$$

Similarly, on solving (3.20), we have

$$v_1(x,y) = t[-2x - x^3(\cot^3\theta - 3\cot^2\theta) + i(12xcot\theta + x^3\cot^3\theta - x^5 cot\theta)]. \tag{3.23}$$

Similarly we can find v_2, v_3 and so on. Required solution is given by

$$u(x,y) = v_0 + v_1 + v_2 + \cdots$$

4. HPM For Solving the Generalized Kdv Equation

Special case 1: When$\theta = \frac{\pi}{4}$, the required solution of $K(2,2)$ reduces to

$$u(x,t) = \frac{x}{1+2t} + 2tx^3 + i[t(12x + x^3 - x^5)] + \cdots$$

Special Case 2: When $\theta = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$ we get the classical $K(2,2)$ equation

$$u_t + (u^2)_x + (u^2)_{xxx} = 0, \tag{3.24}$$

defined in [18] with initial condition (3.17), solution is given as

$$u(x,t) = \frac{x}{1+2t} \tag{3.25}$$

Example 3.3. Now considering the generalized modified KdV equation as

$$u_t + \frac{1}{2}\Delta_x^* u^2 - \Delta_x^{*2} u = 0, \ \ x \in \mathbb{R}, \ t > 0. \tag{3.26}$$

subject to the initial condition

$$u(x,0) = x. \tag{3.27}$$

Proceeding as above homotopy can be written as

$$v_t + p\left[v\frac{\partial v}{\partial x} - \frac{i}{2}v^2 x cot\theta - \frac{\partial^2 v}{\partial x^2} + 2ixcot\theta\frac{\partial v}{\partial x} + (icot\theta + x^2\cot^2\theta)v\right] = 0 \tag{3.28}$$

substituting v and rearranging based on powers of p, we get

$$p^0 \quad : (v_0)_t = 0 \tag{3.29}$$

$$p^1 \quad : (v_1)_t + v_0\frac{\partial v_0}{\partial x} - \frac{i}{2}v_0^2 x cot\theta - \frac{\partial^2 v_0}{\partial x^2} + 2ixcot\theta\frac{\partial v_0}{\partial x} + (icot\theta + x^2\cot^2\theta)v_0 = 0 \tag{3.30}$$

$$p^2 \quad : (v_2)_t + v_0\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_0}{\partial x} - \frac{i}{2}v_0 v_1 x cot\theta - \frac{\partial^2 v_1}{\partial x^2} + 2ixcot\theta\frac{\partial v_1}{\partial x} + (icot\theta + x^2\cot^2\theta)v_1 = 0 \tag{3.31}$$

and so on. Now solving equation (3.29), we get

$$v_0(x,t) = x \tag{3.32}$$

Similarly, from (3.30), we have

$$v_1(x,t) = t[-x - x^3\cot^2\theta + i(\tfrac{1}{2}x^3 cot\theta - 3xcot\theta)], \tag{3.33}$$

from (3.31)

$$v_2(x,t) = \frac{t^2}{2!}\Big[2x - 15xcot^2\theta + \frac{23}{2}x^3\cot^2\theta - \frac{1}{2}x^5\cot^2\theta + x^5\cot^4\theta + i\Big(12xcot\theta - 3x^3 cot\theta + 9x^3\cot^3\theta - \frac{3}{2}x^5\cot^3\theta + x^3\cot^5\theta\Big)\Big] \tag{3.34}$$

similarly, we can find $v_3, v_4, \ldots$, thus we have required solution as

$$u(x,y) = v_0 + v_1 + v_2 + \ldots$$

Special case 1: When $\theta = \frac{\pi}{4}$, the required solution of modified KdV equation is given as:

$$u(x,t) = \left[\frac{x}{1+t} - x^3 t + \frac{t^2}{2!}\left(-15x + \frac{23x^3}{2} + \frac{x^5}{2}\right)\right] + i\left[t\left(\frac{x^3}{2} - 3x\right) + \frac{t^2}{2!}\left(12x + 7x^3 - \frac{3x^5}{2}\right)\right] + \cdots$$

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Special case 2: When $\theta = (2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}$ equation (3.26) converts to modified KdV equation and with the given initial condition (3.27) solution is given as

$$u(x,y) = \frac{x}{1+t}.$$

References

1. J. H. He, *Homotopy perturbation technique*, Comput. Methods Appl. Mech. Engrg. **178**(3-4) (1999), 257-262.
2. J. H. He, *A coupling method of a homotopy technique and perturbation technique for nonlinear problems*, Internat. J. Nonlinear Mech. **35**(1) (2000), 37-43.
3. J. H. He,, *Homotopy perturbation Method: A new nonlinear analytic technique*, Appl. Math. Comput. **135**(1) (2003), 73-79.
4. J. H. He, *New interpretation of homotopy perturbation method*, Internet. J . Modern Physics B **20** (2006), 2561-2568.
5. J. Biazar, H. Ghazvini, *Homotopy perturbation method for solving hyperbolic partial differential equations*, Comput. Math. Appl. , **56**(2) (2008), 453-458.
6. J. H. He, *Application of homotopy perturbation method to nonlinear wave equations*, Chaos Solitons fractals, **26**(3) (2005), 695-700.
7. M. Omidvar, A. Barari, G. Domairry, *Solution of diffusion equations using homotopy perturbation and variational iteration methods*, Trakia Journal of Sciences, **8**(3)(2010), 21-28.
8. J. H. He, *Homotopy perturbation method for solving boundary value problems*, Phys. Lett. A **350**(1-2) (2006), 87-88.
9. M. El-Shahed, *Application of He's homotopy perturabtion method to Volterra's integrodifferential equation*, Int. J. Nonlinear Sci. Numer. Simul. **6**(2) (2005), 163-168.
10. L. Cveticanian, *Homotopy perturbation method for pure nonlinear differential equation*, Chaos solitons fractals **30**(5) (2006), 1221-1230.
11. A. Wazwaz, *A reliable technique for solving the wave equation in an infinite one dimensional medium* , Appl. Math. Comput. **92**(1) (1998), 1-7.
12. Changbum Chun, Hossein Jafri, Yong-ll Kim ,*Numerical method for the wave and nonlinear diffusion equation with the homotopy perturbation method*, Comput. Math. Appl. **57** (2009), 1226-1231.
13. C. S. Gardner, J. M. Green, M. D. Kruskal, and R. M. Miura, *Method for solving the Kortewegde Vries equation*, Phys. Rev. Lett. **19** (1967), 1095-1097.
14. A. A. Soliman, *The modified extended tanh function method for solving Burgers-type equations*, Physica A. **361** (2006), 394–404.
15. M. Wazwaz, *The variational iteration method for rational solutions for KdV,K(2,2), Burgers, and cubic Boussinesq equations*, Journal of Computational and Applied Mathematics **207** (2007), 18-23.

16. H. Tari, D. D. Ganji, M. Rostamian, *Approximate solution of K(2,2), KdV and modified KdV equations by variational iteration method, homotopy perturbation method and homotopy analysis method*, Int. J. Nonlinear Sci. Numer. Simul. **8**(2) (2007), 203-210
17. S. H. Kashkari, *Adomian decomposition method for solving a Generalized Korteweg – De Vries equation with boundary conditions*, Journal of King Abdulaziz University: Science, **23**(2) (2011), 79-90.
18. H. Goodarzian, M. Ghobadi, M.A. Farahabadi, H. Mohammadnezhad and S.S. Hejazi1, *An investigation of nonlinear KdV type equations using HPM and VIM*, Indian Journal of Science and Technology, **4**(8) (2011), 952-956.
19. A. Prasad, S. Manna, A. Mahto, V.K.Singh, *The generalized continuous wavelet transform associated with the fractional Fourier transform*, J. comp. Appl. math. **259** (2014), 660-671.